THE

TERMINAL VELOCITY OF FALL

OF

SMALL SPHERES IN AIR

AT

REDUCED PRESSURES.

A Thesis

Submitted to the Faculty of the Graduate School of the

University of Minnesota

by

Louis Williams McKeehan

in

Partial Fulfillment of the Requirements for the Degree

of

Doctor of Philosophy.

May 8, 1911.

I. Introduction.

The resistance which a fluid offers to a body moving through it increases with the speed, and hence a body falling freely through a fluid attains a limiting velocity called the terminal velocity of fall, when the resistances to motion become equal to the body's resultant weight.

Stokes[1] obtained the expression $6\pi\mu aV$ for the resistance experienced by a spherical body, when all other resistances are negligible compared to that due to the viscosity of the fluid, and when no slipping is supposed to occur at the surface of separation. In this expression, μ is the coefficient of viscosity of the fluid, a the radius of the sphere, and V its velocity through the fluid. The terminal velocity of a freely falling sphere, obtained by putting the above value of the resistance equal to the resultant weight of the sphere, is

$$V = \frac{2}{9}\frac{ga^2(\sigma-\rho)}{\mu},$$

σ being the density of the sphere, and ρ that of the fluid. The exclusion of all but viscous resistance restricts the

[1] G. G. Stokes, Mathematical and Physical Papers, Vol. III., p. 59.

applicability of the formula to very minute spheres, the condition to be fulfilled being that the radius of the sphere must be small compared to $\frac{\mu}{V\rho}$.

Experimental work at atmospheric pressure by Professor John Zeleny and the author[2] has shown this formula to hold for spheres of wax ranging in radius from .002 cm. to .00035 cm., spheres of paraffin, from .002 cm. to .0005 cm., and spheres of mercury, from .001 cm. to .00016 cm., although the measurements on the last named are less accurate than for the other materials, owing to the high reflecting power of their surfaces.

In earlier experiments[*] using natural spores as approximating small spheres, large deviations from the formula given by Stokes were observed, all the spores going too slowly to agree with the formula. It was suggested by Sir Joseph Larmor at the Winnipeg Meeting of the British Association for the Advancement of Science, September,1909,

[2]) Phys. Rev., Vol. XXX., p. 535. Phys. Zeitschr., Bd. 11, s. 78.

[*] Ibid.

that experiments at lower pressures would serve to decide whether or not these deviations were due to the formation of eddies in the air near the not perfectly spherical spores. The force required to maintain these eddies would decrease with the density of the gas, whereas the purely viscous resistance should remain constant, since μ is independent of pressure, at least throughout a very great range[3]. If the large deviations observed with spores were due to this cause, the terminal velocity of fall at a lower pressure should be larger than at atmospheric pressure, and should tend to agree more nearly with that given by the formula of Stokes.

Preliminary experiments using spores at low pressures showed, indeed, an increase in terminal velocity, but one of far too great an amount to be due to the suspected cause, since the velocity became at low pressures several times as great as that calculated by the formula. The reason for this unexpected increase in terminal velocity

---

[3]) Weinstein, Thermodynamik u. Kinetik der Körper, Bd. I., s. 325.

at low pressures becomes clear upon considering more carefully what goes on when a small sphere moves slowly through a gas.

II. Theory.

A paper "On the Velocity of Steady Fall of Spherical Particles through Fluid Medium", by Professor E. Cunningham[4], was communicated to the Royal Society in January, 1910, by Sir Joseph Larmor. An examination of the results there obtained has revealed some inaccuracies. It is believed that the revision here attempted can be most clearly presented by repeating Cunningham's assumptions, and giving the mathematical treatment in full, although this involves a transcription of all that part of the original article which is correct.

"Suppose that a cloud of particles of mass M at a concentration of N particles per unit volume is diffusing through a gas, the mean velocity of the particles being V_0, the mass of the gas molecule being m, and the number of molecules per unit volume being n. It is desired to find the mean force per particle required to maintain the veloc-

[4] Proc. Roy. Soc., Vol. 83A, p. 357, Feb., 1910.

ity V_0.

"It will be assumed in the first instance that the collisions are of the nature of impulses between smooth elastic spheres, and that a collision takes place between a molecule and a particle when the distance apart of their centers is a."

Consider collisions of this type with a single particle, the center of the molecule at collision being within the polar element of area $d\omega = a^2 \sin\theta\, d\theta\, d\phi$ on the surface of the sphere of radius a, (r,θ,ϕ) being spherical polar coordinates having the direction of V_0 as axis.

Let the velocity of the molecule be (u,v,w), and let that of the element $d\omega$ of surface be $(U+V_0\cos\theta, V, W+V_0\sin\theta)$, both referred to rectangular coordinates, u and U being in the direction of the normal to $d\omega$.

The probability of a molecular velocity lying within a range (du,dv,dw) about (u,v,w) is $Ae^{-hm(u^2+v^2+w^2)}du\,dv\,dw$, and that of the velocity of the particle lying within a range (dU,dV,dW) about (U,V,W) is $Be^{-hM(U^2+V^2+W^2)}dU\,dV\,dW$, where $A\left[\frac{\pi}{hm}\right]^{\frac{3}{2}}=1$, and $B\left[\frac{\pi}{hM}\right]^{\frac{3}{2}}=1$.

In order that a collision may take place within the area $d\omega$ in a given time δt, the center of the molecule must, at the beginning of that time lie within a cylindrical volume of base $d\omega$ and of height $(U+V_\circ \cos\theta-u)\delta t$, with the restriction that the relative velocity $(U+V_\circ \cos\theta-u)$ must be positive.

The impulse imparted to the particle by the impact of one molecule is

$$I_u=\frac{2Mm}{M+m}\left(U+V_\circ \cos\theta-u\right)$$

normal to the surface $d\omega$. The impulse in the direction of $V_\circ$ is $I_\circ = I_u \cos\theta$. The probable impulse per particle per unit time in the direction of $V_\circ$ is the integral of the impulse per impact in the direction of $V_\circ$, multiplied by the probabilities of velocities within the limits (du,dv,dw) about (u,v,w), and (dU,dV,dW) about (U,V,W), multiplied by the number of molecules in the cylindrical volume $(U+V_\circ \cos\theta-u)d\omega$, that is

$$\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int n\left[Ae^{-hm(u^2+v^2+w^2)}du\,dv\,dw\right]\left[Be^{-hM(U^2+V^2+W^2)}dU\,dV\,dW\right]\left[I_u\cos\theta\right]\left[U+V_\circ\cos\theta-u\right]d\omega\,,$$

the limits for v,w,V,W,θ, and ϕ being unrestricted, but those for u and U being conditioned by the equation

$$U+V_\circ\cos\theta-u \not< 0.$$

Substituting the values of I_u and $d\omega$, and integrating with respect to v,w,V,W, and ϕ, we get

$$4ABna^2\frac{\pi^2}{h^2(M+m)}\iiint e^{-h(mu^2+MU^2)}\left[U+V_0\cos\theta-u\right]^2\sin\theta\cos\theta\,d\theta\,du\,dU,$$

the limits for θ being 0 and π, those for u and U as given above. This is integrable only if V_0 is small compared to the mean value of U-u. Making this assumption, and calling the integral K_1, we have

$$K_1 = -\,-\iiint e^{-\frac{h\left[mM\alpha^2+(M+m)^2\beta^2\right]}{M+m}}\left[\alpha+V_0\cos\theta\right]^2\sin\theta\cos\theta\,d\theta\,d\alpha\,d\beta,$$

where $\alpha = U-u$, $d\alpha = -du$,

and $\beta = \frac{MU+mu}{M+m}$, $d\beta = dU$.

Neglecting the mean value of U with respect to the mean value of u, the limits become,

for u, $-\infty$ and $V_0\cos\theta$,

for α, ∞ and $-V_0\cos\theta$,

and for β, ∞ and $-\infty$.

Integrating with respect to β between these limits

$$K_1 = -\,-\sqrt{\frac{\pi}{h(M+m)}}\int_0^{\pi}\int_{-V_0\cos\theta}^{\infty} e^{-\frac{hmM\alpha^2}{M+m}}\left[\alpha+V_0\cos\theta\right]^2\sin\theta\cos\theta\,d\theta\,d\alpha.$$

Neglecting $V_0\cos\theta$ compared with the mean value of α, changing the lower limit for α to 0,

$$K_1 = -\,-\sqrt{\frac{\pi}{h(M+m)}}\int_0^{\pi}\int_0^{\infty} e^{-\frac{hmM\alpha^2}{M+m}}\left[\alpha+V_0\cos\theta\right]^2\sin\theta\cos\theta\,d\theta\,d\alpha.$$

Integrating first with respect to θ ,

$$\int_0^{\pi}\left[\alpha+V_0\cos\theta\right]^2\sin\theta\cos\theta\,d\theta$$

$$= \frac{4\alpha V_0}{3}.$$

Substituting this value

$$K_1 = -\sqrt{\frac{\pi}{h(M+m)}} \cdot \frac{4V_0}{3}\int_0^\infty \alpha e^{-\frac{hmM\alpha^2}{M+m}} d\alpha$$

$$= \frac{2}{3}V_0\sqrt{\frac{\pi(M+m)}{h^3m^2M^2}}.$$

The mean force per particle required to maintain the velocity V_0 is, then,

$$F_1' = 4ABna^2\frac{\pi^3}{h^2(M+m)} \cdot \frac{2}{3}V_0\sqrt{\frac{\pi(M+m)}{h^3m^2M^2}},$$

or, neglecting m with respect to M,

$$F_1' = \frac{8}{3}a^2nV_0\sqrt{\frac{\pi m}{h}}.$$

In this expression V_0 is the difference between the mean velocity of the particles and the mass velocity of the gas at their surfaces. In the derivation of Stokes's formula it is assumed that there is no slip at the surface of the particle, so that the mass velocity of the gas is the same as that of the particle, and the expression above becomes equal to zero. Assuming instead that the mass velocity of the gas at the surface of the particle is k_1V, where k_1 is a factor less than unity, and V is the velocity of the particle referred to a fixed system of coordinates, $V_0 = V - k_1V$, and since the force required to maintain a velocity k_1V of a spherical layer of gas of radius a is, by the ordinary theory equal to $6\pi\mu ak_1V$, where μ is the coefficient of viscosity, the force required to maintain the velocity V of the particle is the same, or

$$X_1 = 6\pi\mu a k_1 V = \frac{8}{3}a^2 n(V - k_1 V)\sqrt{\frac{\pi m}{h}} ,$$

from which

$$k_1 = \frac{4an\sqrt{\frac{\pi m}{h}}}{4an\sqrt{\frac{\pi m}{h}} + 9\pi\mu} = \left[1 + \frac{9\mu\sqrt{\pi h m}}{4anm}\right]^{-1}.$$

Substituting[5] $\sqrt{hm} = \frac{2}{\bar{c}\sqrt{\pi}}$, $nm = \rho$, and $\frac{\mu}{\rho\bar{c}} = \frac{l}{3}$,

where $\bar{c}$ is the mean velocity of the gas molecule, l is the mean free path, and ρ is the density , we get

$$k_1 = \left[1 + \frac{3l}{2a}\right]^{-1} ,$$

$$X_1 = 6\pi\mu a V\left[1 + 1.5\frac{l}{a}\right]^{-1} .$$

In case gravity is the only force acting, the effective weight of the particle is $\frac{4}{3}\pi a^3 g(\sigma - \rho)$ where σ is the density of the particle. Equating this to

---

[5] Jeans, Dynamical Theory of Gases, p. 250. One of the substitutions was made incorrectly in a note published in the Physical Review, Vol. XXXII., p. 341, March, 1911, so that the numerical value of the constant multiplier of $\frac{l}{a}$ is there given wrongly. This mistake was pointed out by Prof. R. A. Millikan, who kindly read a part of the analysis here given. The same mistake appears in Jeans work cited above.

X_1, and solving for V, we get the limiting velocity of steady fall

$$V = \frac{2a^2g(\sigma-\rho)}{9\mu}\left[1 + 1.5\frac{l}{a}\right],$$

which exceeds that given by the ordinary formula by an amount inversely proportional to the density of the gas.

Suppose, on the other hand, that the impacts of the molecules on the particle are not elastic collisions, but that each molecule enters the surface layer of the particle (whether the solid material or a layer of condensed gas is for the present purpose immaterial), and emerges again from the same area $d\omega$ on which it impinged, but with a velocity independent, in direction and amount, of the relative velocity of the particle and molecule before collision.

Taking the same system of rectangular coordinates as before, the component velocities of $d\omega$ are ($U+V_0\cos\theta$, V, $W+V_0\sin\theta$); the component velocities of the molecule just before impact are (u,v,w), and just after emergence are ($-u+U+V_0\cos\theta$, $v+V$, $w+W+V_0\sin\theta$). The sign of u has been changed to represent the fact that the molecule, which approached $d\omega$ when its velocity was u, now recedes from that surface. The component velocities (u,v,w) in the last expressions given above are not necessarily the same as (u,v,w) for that particular molecule before impact; but, considering the whole number of impacts, a colliding molecule can always be found which had these component velocities when it impinged on $d\omega$. This rests on the assumption that (u,v,w) on emergence follow the same law of distribution as (u,v,w) at impact.

The changes in velocity of the molecule along the three axes due to penetration and emergence, or so-called inelastic impact are

$$(-u+U+V_0\cos\theta)-u,\quad (v+V)-v,\quad (w+W+V_0\sin\theta)-w,$$

or $(U+V_0\cos\theta-2u)$, V, $(W+V_0\sin\theta)$.

The impulses imparted to the molecule in these three directions are

$$I_u = \frac{Mm}{M+m}\left(U+V_0\cos\theta-2u\right),$$

$$I_v = \frac{Mm}{M+m}V,$$

$$I_w = \frac{Mm}{M+m}\left(W+V_0\sin\theta\right).$$

The impulse in the direction of V_0 is

$$\begin{aligned} I_0 &= I_u\cos\theta + I_w\sin\theta \\ &= \frac{Mm}{M+m}\left(U\cos\theta + V_0\cos^2\theta - 2u\cos\theta + W\sin\theta + V_0\sin^2\theta\right) \\ &= \frac{Mm}{M+m}\left(U\cos\theta + W\sin\theta + V_0 - 2u\cos\theta\right). \end{aligned}$$

This replaces the expression

$$I_0 = \frac{2Mu}{M+m}\left(U\cos\theta + V_0\cos^2\theta - u\cos\theta\right)$$

in the preceding analysis, the other terms being the same as before. The integral expressing the mean value of the force required to maintain the velocity of a single particle is $F_z' =$

$$-\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int u\left[Ae^{-hm(u^2+v^2+w^2)}du\,dv\,dw\right]\left[Be^{-hM(U^2+V^2+W^2)}dU\,dV\,dW\right]\frac{Mm}{M+m}\left[U\cos\theta+W\sin\theta+V_0-2u\cos\theta\right]\left[U+V_0\cos\theta-u\right]d\omega.$$

Integrating with respect to v,w,V, andϕ,

$$F_z' = -2a^2\left(\frac{Mm}{M+m}\right)\left(\frac{\pi}{hm}\right)\left(\frac{\pi}{hM}\right)^{\frac{1}{2}}AB\pi\iiiint e^{-h(mu^2+MU^2+MW^2)}\left[U\cos\theta+W\sin\theta+V_0-2u\cos\theta\right]\left[U+V_0\cos\theta-u\right]\sin\theta\,d\theta\,du\,dU\,dW.$$

Noting that $\int_{-\infty}^{\infty}e^{-hMW^2}W\,dW = 0$,

and that $\int_{-\infty}^{\infty}e^{-hMW^2}dW = \left(\frac{\pi}{hM}\right)^{\frac{1}{2}}$,

$$F_z' = -\frac{2\pi^2a^2u}{h^2(M+m)}AB\iiint e^{-h(mu^2+MU^2)}\left[U\cos\theta+V_0-2u\cos\theta\right]\left[U+V_0\cos\theta-u\right]\sin\theta\,d\theta\,du\,dU.$$

Let the integral in this expression be denoted by K_z.

$$K_z = \iiint_0^{\pi} e^{-h(mu^2+MU^2)}du\,dU\left[\left(U^2+V_0^2+2u^2-3Uu\right)\cos\theta\sin\theta+\left(UV_0-2uV_0\right)\cos^2\theta\sin\theta+\left(UV_0-uV_0\right)\sin\theta\right]d\theta$$

$$= V_0\iint e^{-h(mu^2+MU^2)}\left[0+\tfrac{8}{3}U-\tfrac{10}{3}u\right]du\,dU$$

$$= \frac{2V_0}{3}\iint e^{-h(mu^2+MU^2)}\left[4U-5u\right]du\,dU$$

Considering the fact that the mean value of u is very large compared to the mean value of U, we can assign the limits for these variables, then

$$K_z = \frac{2V_0}{3}\int_{-\infty}^{\infty}e^{-hMU^2}\int_0^{\infty}e^{-hmu^2}\left[4U-5u\right]du$$

$$= \frac{2V_0}{3}\int_{-\infty}^{\infty}e^{-hMU^2}\left[2U\sqrt{\frac{\pi}{hm}}-\frac{5}{2}\cdot\frac{1}{hm}\right]dU$$

$$= -\frac{5V_0}{3hm}\int_{-\infty}^{\infty} e^{-hMU^2}\,dU = -\frac{5V_0}{3}\sqrt{\frac{\pi}{h^3 m^2 M}}\,.$$

Substituting,

$$F_2' = \frac{2\pi^{\frac{3}{2}} a^2 n AB}{h^2(M+m)}\cdot\frac{5V_0\,\pi^{\frac{1}{2}}}{3h^{\frac{3}{2}} m M^{\frac{1}{2}}},\quad A = \frac{\pi^{\frac{1}{2}}}{h^{\frac{1}{2}} m^{\frac{1}{2}}},\quad B = \frac{\pi^{\frac{1}{2}}}{h^{\frac{1}{2}} M^{\frac{1}{2}}}\cdot\ \frac{10}{3}a^2 n V_0 \frac{M m^{\frac{1}{2}}\pi^{\frac{1}{2}}}{h^{\frac{1}{2}}(M+m)}$$

$$= \frac{10}{3}a^2 n V_0\sqrt{\frac{\pi M^2 m}{h(M+m)^2}} = \frac{10}{3}a^2 n V_0\sqrt{\frac{\pi m}{h}}$$

$$= \frac{5}{4}F_1'\,.$$

This gives, using the same notation as in the elastic case

$$X_2 = 6\pi\mu a k_2 V = \frac{10}{3}a^2 n(V - k_2 V)\sqrt{\frac{\pi m}{h}}\,.$$

$$k_2 = \left[1 + \frac{4}{5}\cdot\frac{3l}{2a}\right]^{-1} = \left[1 + 1.2\frac{l}{a}\right]^{-1}.$$

$$X_2 = 6\pi\mu a V\left[1 + 1.2\frac{l}{a}\right]^{-1},$$

$$V = \frac{2a^2 g(\sigma-\rho)}{9\mu}\left[1 + 1.2\frac{l}{a}\right].$$

If it be assumed that a fraction f of the molecular impacts on the surface of the particle are elastic collisions, and the remaining fraction (1-f) are inelastic, in the sense explained above, the mean impulse per second per particle due to the first kind is fF_1, and that due to the second kind is $(1-f)F_2$. Their sum is

$$F_3' = fF_1' + (1-f)F_2' = \left[f + (1-f)\frac{5}{4}\right]F_1' = \frac{5-f}{4}F_1'\,.$$

From this $\quad k_3 = \left[1 + 1.5\frac{l}{a}\left(\frac{4}{5-f}\right)\right]^{-1}$

and $\quad V = \frac{2a^2 g(\sigma-\rho)}{9\mu}\left[1 + 1.5\frac{l}{a}\left(\frac{4}{5-f}\right)\right]\quad ,$

which formula might be used for the determination of f, when all the other terms in it are known. Cunningham gets

$$V = \frac{2a^2g(\sigma-\rho)}{9\mu}\left[1 + 1.63\frac{l}{a}\left(\frac{1}{2-\rho}\right)\right].$$

Suppression of the component velocities (U,V,W) of the particle has no effect on the result obtained, and simplifies the analysis. In either case the volume throughout which the integration is to be effected becomes

$$(V_0 \cos\theta - u)d\omega.$$

In the elastic case the impulse per impact in the direction of V_0 becomes

$$I_0 = \frac{2Mm}{M+m}\left(V_0\cos^2\theta - u\cos\theta\right),$$

so that

$$F_1 = -\int\!\!\int\!\!\int\!\!\int\!\!\int\left[Ae^{-hm(u^2+v^2+w^2)}du\,dv\,dw\right]\frac{2Mmn}{M+m}\left(V_0\cos^2\theta - u\cos\theta\right)\left(V_0\cos\theta - u\right)d\omega$$

$$= -\frac{4\pi^2a^2MnA}{h(M+m)}\int\!\!\int e^{-hmu^2}\left(V_0\cos\theta - u\right)^2\sin\theta\cos\theta\,d\theta\,du.$$

$$K_1 = \int_0^\infty\!\!\int_0^\pi e^{-hmu^2}\left(V_0\cos\theta - u\right)^2\sin\theta\cos\theta\,d\theta\,du$$

$$= -\frac{4V_0}{3}\int_0^\infty u e^{-hmu^2}du = -\frac{2V_0}{3hm}.$$

$$F_1 = \frac{4\pi^2a^2MnA}{h(M+m)}\cdot\frac{2V_0}{3hm} = A\frac{\pi^{\frac{3}{2}}}{h^{\frac{3}{2}}m^{\frac{3}{2}}}\cdot\frac{8}{3}a^2nV_0\sqrt{\frac{\pi m M^2}{h(M+m)}}$$

$$= \frac{8}{3}a^2nV_0\sqrt{\frac{\pi m}{h}},$$

as before.

In the inelastic case the impulse per impact in the direction of V_0 becomes

$$I_0 = \frac{Mm}{M+m}\left(V_0\cos^2\theta - 2u\cos\theta + V_0\sin^2\theta\right),$$

so that

$$F_2' = -\iiiint \left[A e^{-hm(u^2+v^2+w^2)} du\,dv\,dw\right] \frac{Mmn}{M+m} \left[V_0 - 2u\cos\theta\right]\left[V_0\cos\theta - u\right] d\omega$$

$$= -\frac{2\pi^2 a^2 MnA}{h(M+m)} \iint e^{-hmu^2} \left[V_0 - 2u\cos\theta\right]\left[V_0\cos\theta - u\right] \sin\theta\, d\theta\, du.$$

$$K_2 = \int_0^\infty \int_0^\pi e^{-hmu^2} \left[V_0^2\cos\theta - 2uV_0\cos^2\theta - uV_0 + 2u^2\cos\theta\right] \sin\theta\, d\theta\, du$$

$$= \int_0^\infty e^{-hmu^2} \left(0 - \frac{4V_0 u}{3} - 2V_0 u\right) du = -\frac{10}{3} V_0 \int_0^\infty u e^{-hmu^2} du = -\frac{5V_0}{3hm}.$$

$$F_2' = \frac{2\pi^2 a^2 MnA}{h(M+m)} \cdot \frac{5V_0}{3hm}$$

$$= A\frac{\pi^{\frac{3}{2}}}{h^{\frac{3}{2}} m^{\frac{3}{2}}} \cdot \frac{10}{3} a^2 n V_0 \sqrt{\frac{\pi M^2 m}{h(M+m)^2}} = \frac{10}{3} a^2 n V_0 \sqrt{\frac{\pi m}{h}}$$

$$= \frac{5}{4} F_1'$$

as before.

Another assumption leading to integrable equations can be made regarding the impacts, and may be considered as a limiting case. Let us assume that after striking the surface of a particle a molecule is only able to leave it again when moving in the direction of the normal to the surface. In order to keep the average translatory energy of the molecules unaltered we must make the absolute normal velocity after impact equal to $-u\sqrt{3}$ instead of $-u$, as in the preceding cases. We then have as before the component velocities of $d\omega$,

$(U+V_0\cos\theta)$, V, $(W+V_0\sin\theta)$,

and the component velocities of the molecule just before impact (u,v,w).

But just after impact the component velocities become

$$(-u\sqrt{3}+U+V_\circ\cos\theta),\ V,\ (W+V_\circ\sin\theta).$$

The changes in velocity of the molecule are, therefore,

$$(-u\sqrt{3}+U+V_\circ\cos\theta)-u,\ V-v,\ (W+V_\circ\sin\theta)-w,$$

or $\left[U+V_\circ\cos\theta-u(1+\sqrt{3})\right]$, $\left[V-v\right]$, $\left[W+V_\circ\sin\theta-w\right]$,

and the impulses imparted to the molecule in these three directions are

$$I_u=\frac{Mm}{M+m}\left[U+V_\circ\cos\theta-u(1+\sqrt{3})\right],$$

$$I_v=\frac{Mm}{M+m}\left[V-v\right],$$

$$I_w=\frac{Mm}{M+m}\left[W+V_\circ\sin\theta-w\right].$$

The impulse in the direction of $V_\circ$ is

$$I_\circ=I_u\cos\theta+I_w\sin\theta.$$

Substituting, $$I_\circ=\frac{Mm}{M+m}\left[U\cos\theta+V_\circ\cos^2\theta-u(1+\sqrt{3})\cos\theta+W\sin\theta+V_\circ\sin^2\theta-w\sin\theta\right]$$

$$=\frac{Mm}{M+m}\left[U\cos\theta+W\sin\theta+V_\circ-u(1+\sqrt{3})\cos\theta-w\sin\theta\right].$$

The mean force required to maintain the velocity of the particle is, then,

$$F_\circ=-\iiiint\!\!\iiint n\left[Ae^{-hm(u^2+v^2+w^2)}du\,dv\,dw\right]\left[Be^{-hM(U^2+V^2+W^2)}dU\,dV\,dW\right]\frac{Mm}{M+m}\left[U\cos\theta+W\sin\theta+V_\circ-u(1+\sqrt{3})\cos\theta-w\sin\theta\right]\left[U+V_\circ\cos\theta-u\right]d\omega.$$

Integrating with respect to v, V, and ϕ.

$$F_\circ=-2\pi a^2\frac{Mmn}{M+m}\left(\frac{\pi}{hm}\right)^{\frac{1}{2}}\left(\frac{\pi}{hM}\right)^{\frac{1}{2}}AB\iiiint\!\!\int e^{-h(mu^2+mw^2+MU^2+MW^2)}\left[U\cos\theta+W\sin\theta+V_\circ-u(1+\sqrt{3})\cos\theta-w\sin\theta\right]\left[U+V_\circ\cos\theta-u\right]\sin\theta\,d\theta\,du\,dw\,dU\,dW.$$

Since $\displaystyle\int_{-\infty}^{\infty}We^{-hMW^2}dW=0$, $\displaystyle\int_{-\infty}^{\infty}e^{-hMW^2}dW=\left(\frac{\pi}{hM}\right)^{\frac{1}{2}}$,

$$F_4' = -2\pi a^2 \frac{Mmn}{M+m}\left(\frac{\pi}{hm}\right)^{\frac{1}{2}}\left(\frac{\pi}{hM}\right) AB \iiiint e^{-h(mu^2+mw^2+MU^2)} \left[U\cos\theta + V_0 - u(1+\sqrt{3})\cos\theta - w\sin\theta\right]\left[U+V_0\cos\theta - u\right]\sin\theta\, d\theta\, du\, dw\, dU.$$

Also $\displaystyle\int_{-\infty}^{\infty} w e^{-hmw^2} dw = 0$, $\displaystyle\int_{-\infty}^{\infty} e^{-hmw^2} dw = \left(\frac{\pi}{hm}\right)^{\frac{1}{2}}$,

$$F_4' = -2\pi a^2 \frac{Mmn}{M+m}\left(\frac{\pi}{hm}\right)\left(\frac{\pi}{hM}\right) AB \iiint e^{-h(mu^2+MU^2)} \left[U\cos\theta + V_0 - u(1+\sqrt{3})\cos\theta\right]\left[U+V_0\cos\theta - u\right]\sin\theta\, d\theta\, du\, dU.$$

The integral in this expression

$$K_0 = \iiint_0^{\pi} e^{-h(mu^2+MU^2)} du\, dU \left[\left\{U^2+V_0^2+u^2(1+\sqrt{3}) - uU(2+\sqrt{3})\right\}\cos\theta\sin\theta + \left\{UV_0 - uV_0(1+\sqrt{3})\right\}\cos^2\theta\sin\theta + \left\{UV_0 - uV_0\right\}\sin\theta\right] d\theta$$

$$= V_0 \iint e^{-h(mu^2+MU^2)} \left[0 + \tfrac{8}{3}U - \tfrac{8+2\sqrt{3}}{3}u\right] du\, dU$$

$$= \frac{2V_0}{3} \iint e^{-h(mu^2+MU^2)} \left[4U - (4+\sqrt{3})u\right] du\, dU = \frac{2V_0}{3}\int_{-\infty}^{\infty} e^{-hMU^2} dU \int_0^{\infty} e^{-hmu^2}\left[4U-(4+\sqrt{3})u\right] du$$

$$= -\frac{4+\sqrt{3}}{3}\cdot\frac{V_0}{hm}\int_{-\infty}^{\infty} e^{-hMU^2} dU = -\frac{(4+\sqrt{3})V_0}{3}\sqrt{\frac{\pi}{h^3m^2M}} .$$

$$F_4' = 2\pi a^2 \frac{Mmn}{M+m}\left(\frac{\pi}{hm}\right)\left(\frac{\pi}{hM}\right) AB \frac{(4+\sqrt{3})V_0}{3}\sqrt{\frac{\pi}{h^3m^2M}} = \frac{4+\sqrt{3}}{4}\cdot\frac{8}{3}a^2 n V_0 \sqrt{\frac{\pi m}{h}}$$

$$= \frac{4+\sqrt{3}}{4} F_1' .$$

Substituting this value for F_4 ,

$$k_4 = \left[1 + \frac{4}{4+\sqrt{3}}\cdot\frac{3l}{2a}\right]^{-1} = \left[1 + 1.05\frac{l}{a}\right]^{-1} ,$$

and $\displaystyle V = \frac{2a^2 g(\sigma-\rho)}{9\mu}\left[1 + 1.05\frac{l}{a}\right]$.

The results of the preceding analysès may be summarized as follows.

The velocity of fall of a small sphere in a gas is given by the equation

$$V = V_s\,(1+A\tfrac{l}{a}),$$

wherein V_s is the velocity given by Stokes's formula. If the molecular impacts on the surface of the sphere are all elastic collisions A=1.5 . If they are all inelastic collisions $A=1.2$. If a fraction f are of the former, and the rest of the latter type $A=1.5(\frac{4}{5-f})$. If all the impinging molecules emerge normally to the surface $A=1.05$.

Since the mean free path l of the gas molecule is inversely proportional to the pressure p of the gas, an alternative formula

$$V = V_s\left(1+\frac{B}{ap}\right)$$

can be used. Calculating the value of l from the viscosity of air at 20° C., ($\mu = 1833\times10^{-7}$)[6], and $\bar{c} = 463\times10^{2}$ at the same temperature, we get when p is expressed in mm. of mercury, for

A	B
1.0	.0075
1.05	.0078
1.2	.0090
1.5	.0112

The terminal velocity of fall of small spheres should thus increase enormously at very low pressures, and should then become proportional to the radius of the sphere, and not to the square of the radius as at higher pressures. In figures 1 and 2 are plotted as abscissa the reciprocal of the computed velocity, that is, the time

[6] R. A. Millikan, Phil. Mag., VI, 19, p. 215, Feb., 1910. He gives $\mu = 1863\times10^{-7}$ at 26° C.

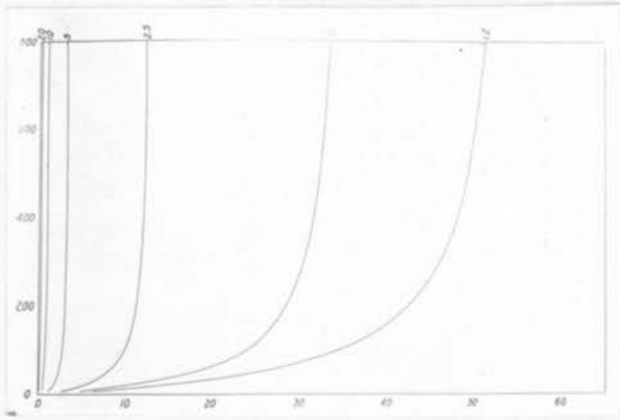


Figure 1.

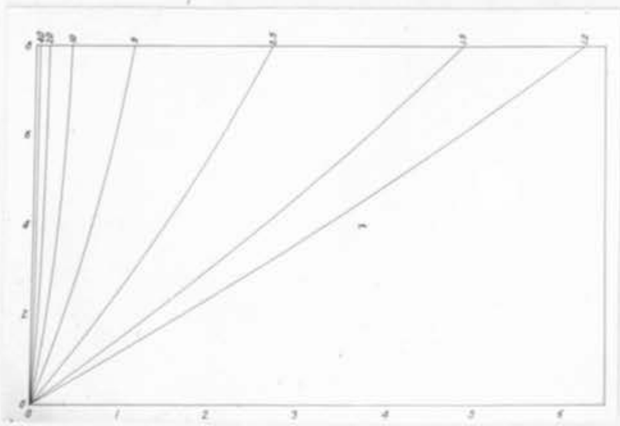


Figure 2.

required to fall one centimeter at this velocity, and as ordinate the pressure in mm. of mercury. The temperature assumed is 20° C., and the value of the constant chosen is $A = 1.0$. Six curves are drawn in figure 1 for different values of the radius, the value for each curve being given in $cm \times 10^{-4}$ in the upper margin. Figure 2 shows on a much larger scale the part near the origin omitted from figure 1. The close approach to a linear variation of time of fall with pressure should here be noted. It is found upon computation that the criterion for steady motion given in the second paragraph of the introduction is fulfilled by all the sizes shown at all the pressures , except for the curve for radius .004 at more than 300 mm. pressure.

The assumption is made that μ has the same value at all pressures. This is required by the kinetic theory of gases and it seems likely that the values of μ obtained experimentally at low pressures are lower than at atmospheric pressure only because slip is neglected. For this reason the pressure at which μ shows a marked decrease depends on the method used in determining it, and upon the dimensions of the apparatus employed .

III. Experiments.

1. Preparation of Material.

Minute spheres of wax were made from molten wax by an atomizer[7], and were collected by allowing the cloud of spheres so formed to settle on sheets of paper in a closed space about 40×40×60 cm. The range of sizes obtained could be varied within certain limits by regulating the size and position of the openings of the atomizer, and by varying the temperature of the wax. All the sizes used in the experiments were produced simultaneously.

2. Measurements.

In order to find the value of B in the formula

$$V = \frac{2a^2 g(\sigma - \rho)}{9\mu}\left(1 + \frac{B}{ap}\right),$$

the values of σ, the density of a sphere, a, its radius, V, its terminal velocity of fall, and p, the pressure in mm. of mercury, were determined for over five hundred spheres of wax.

3. Density.

The density, σ, of the wax was 1.058, as previously determined for a large piece.

[7] The same method was used in the earlier work cited.

4. Terminal Velocity of Fall.

The method of getting the terminal velocity of fall, V, was the same as that used in the previous experiments, and consisted in measuring the total time required to fall a known distance of about 31 cm.

Referring to figure 3: drawing I is a vertical section through the apparatus; II is a plan of the same with the cover removed; III is a horizontal section through the tube in which the spheres fell; IV is an enlarged view of the lower left hand corner of I; V and VI are details of parts not completely shown elsewhere; VII and VIII are plans and sections of the two mechanisms used for releasing spheres. The lettering of corresponding parts is the same in the first six drawings.

In releasing spheres by the method shown in drawing VII, a quantity of them several mm^3. in volume was placed in the cloth-bottomed trough E, which slipped tightly into a slot in the frame G. A solenoid A drew in the soft-iron armature B, and by it moved the sliding plate C across the short open tube H, soldered to the bottom of the fall box just under E. The sliding plate carried a flexible pointed

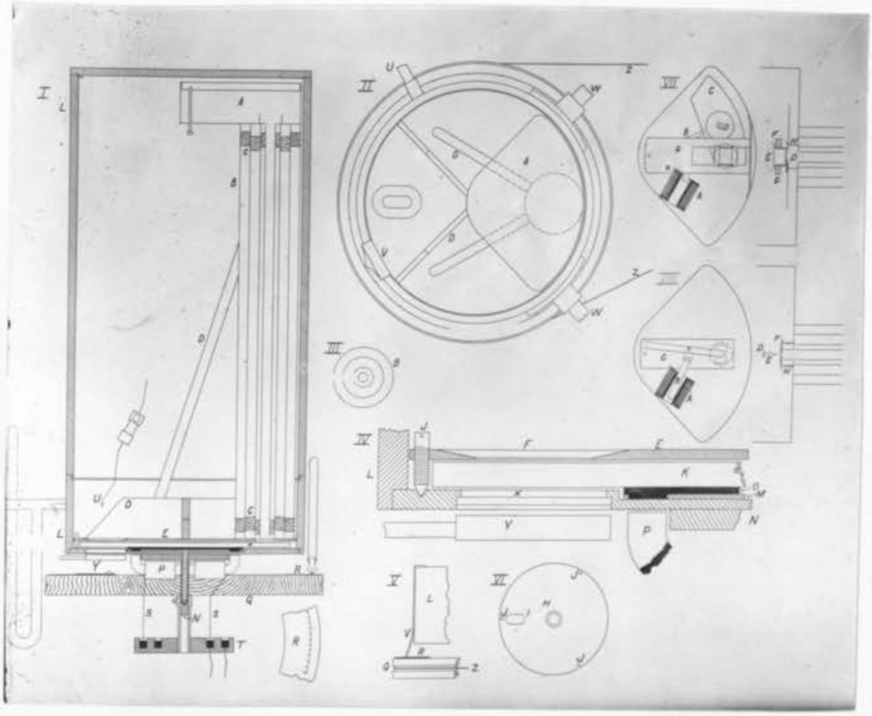


Figure 3.

spring D projecting above a small hole in its center. The motion of C was stopped by a peg, not shown, when D reached the middle of the bottom of the trough E, against which it scratched lightly. The vibration produced by this scratch sifted some spheres through the cloth, and they then fell through the small hole in C, through H, and into the fall tube, B.

In the method shown in drawing VIII the spheres were left on the paper used in collecting them. A piece of this was put at F, with the sphere-coated side underneath, just over a small hole in the frame G, which fit tightly in the tube H. The solenoid A, by means of the armature B drew a brass block C from beneath a long metal strip D, which sprung downward, rapping the point E against the upper surface of the paper just over the hole in G. The blow was not sufficient to puncture the paper, but it set some of the spheres free, and they then fell through H into the fall tube.

In the fall tube B the spheres, released by either of the methods described, fell through the innermost of four concentric brass tubes, designed to prevent air currents.

The outer tubes were pierced at one point, as shown in drawing III, to facilitate changes of pressure in the spaces between the wooden rings placed near the ends. The three inner tubes were entirely supported by these rings, and were not in contact with any other metal parts. The apparatus was used in a room with double walls and no windows, which provided an excellently steady temperature.

The outer tube was soldered at the top to the fall box A, and at the bottom to braces D,D, and to the brass plate E. This plate was supported on three leveling screws J, drawing IV, one of which rested in a socket, fixing the relative angular position of the fall tube and the outer case L. The plate was pierced by two holes, one a narrow radial slot just under the fall tube, and the other a window, 180° from the slot, for viewing the spheres after they had fallen. A plate H, drawing VI, similar to E, but without any fall tube, was substituted for it when measuring the spheres. This provided more room for the measuring microscope.

The surface for receiving the spheres was that of a plate glass disc K, cemented to a circular brass plate M, and rotating close under the plate E on a pivot in the bearing N. On a diameter of the plate M was imbedded a soft-iron armature O with thickened ends. The poles of this armature were directly over the poles of an electro-magnet P carried on a wooden wheel Q, which was rotated about N by a chronograph motor through a belt Z. The current for P was supplied through wires S,S, dipping in circular troughs of mercury in the ebonite block T. By this device the bearing for the plate K was nowhere exposed to atmospheric pressure, and required no packing.

At the instant of releasing the spheres the wheel Q was set in steady rotation by putting it in gear with the chronograph motor, and as the spheres arrived successively at the bottom of the fall tube, they were received at different angular positions on the surface of the plate K. A chronograph pen connected in series with a seconds pendulum, rested on a paper strip R attached to Q, and marked each second as the wheel rotated. This time record is shown in plan just to the right of T, and in drawing II, for a different speed of rotation. After completing the experiment and stopping the rotation of Q, an index V, which fit over the edge of the case L, was placed exactly 180° from the place the pen had ceased to mark. When this index pointed to any mark on the pen record, as Q was rotated, the spheres which had fallen on K just as that mark was being made would be 180° from the bottom of the fall tube; that is, they would be in a position to be seen through the window at F. By counting seconds from the beginning of the pen record to the index, the elapsed time of fall was given directly.

This is a longer time than that which would be required if the sphere had started to fall with its terminal velocity, instead of starting from rest. The correction to be applied for this reason can be derived as follows.

The differential equation of the variable velocity v is

$$\frac{dv}{dt} + kv = g \quad ,$$

where $$k = \frac{9\mu}{2a^2\,(\sigma-\rho)\left(1+\frac{B}{a\rho}\right)}$$

The solution for a motion starting from rest, v=0, when t=0, is

$$v = \frac{g}{k}\left(1-e^{-kt}\right) .$$

Let T be the observed time of fall. Then the velocity finally attained in the apparatus is

$$V' = \frac{g}{k}\left(1 - e^{-kT}\right).$$

The exponential term is negligible compared to unity. To show this take as a case giving it the largest possible value for the experiments actually performed, a sphere of radius a = .003 cm. at a pressure of .3 mm. of mercury, for which T = .3 sec. Using a value of B determined later

$$e^{-kT} = .00004 .$$

Therefore we can put

$$V' = \frac{g}{k} = V ,$$

the terminal velocity to be found. If s is the length of the fall, the time required to fall uniformly through this distance at the velocity V would have been

$$T' = \frac{s}{V'} = \frac{sk}{g} ,$$

from which $s = \frac{gT'}{k}.$

Also, by integrating the variable velocity v throughout the time of fall, T,

$$s = \int_0^T v\,dt = \frac{g}{k}\left\{T - \frac{1 - e^{-kT}}{k}\right\} ;$$

again disregarding the term e^{-kT} in comparison with unity,

$$s = \frac{g}{k}\left\{T - \frac{1}{k}\right\} ;$$

equating the two values for s gives

$$T' = T - \frac{1}{k} .$$

Putting for k its value $\frac{gT'}{s}$,

$$T' = T - \frac{s}{gT'} .$$

To a first approximation

$$T' = T - \frac{s}{gT} .$$

that is, the correction to be applied to the observed times of fall to give the times of fall at the terminal velocity depends only on the length of the fall and the time of fall. The following table gives the corrections to the observed time of fall in the apparatus used.

Observed time. Seconds.	Correction. Seconds.
.50	-.06
1.00	-.03
2.00	-.02
3.00	-.01
6.00	-.01
7.00	-.00

These corrections have been applied to all times of fall less than seven seconds.

5. Radius of Spheres.

The radius, a, of each sphere was determined by measuring its diameter from two to eight times, the majority of the spheres being measured six times. An improved microscope plate micrometer[8] was used except in experiments II., 1, 2, and 3, and for 14 spheres in other experiments, too large to be measured by the plate micrometer. These excepted spheres were measured by an ocular micrometer, by Zeiss. The microscope field was illuminated by diffused light reflected from the mirror Y (drawing IV, figure 3) through the window X and disc K. Test measurements using different magnifying powers gave concordant results for all the sizes used, and it appears that the measurements of radii, even for the smallest spheres, are not affected by constant errors due to diffraction, of an amount exceeding a small fraction of a wave length of light. Some difficulty in the measurement of the smaller spheres was experienced on account of tremors of the pier on which the apparatus stood. The trouble was eliminated by measuring diameters after midnight, when heavy traffic on the streets had ceased. The method of making exterior contact of the cross-hair and the circular image of a sphere was found the most convenient for rapid measurement. A correction

[8] John Zeleny and L. W. McKeehan, Phys. Rev., Vol. XXXII., p. 530.

for the width of the cross-hair had, therefore, to be applied to all diameters measured. The amount of this correction was determined by using other forms of contact for comparaison, with other objects than spheres.

6. Pressure.

The pressure inside the case L, which was of brass about 5 mm. thick, was reduced by a water pump for intermediate, and by a Gaede mercury pump for lower, pressures. The pumps were attached at U, figure 3. The window X, figure 3, drawing IV, was of plate glass set in paraffin, making a permanently air-tight joint. The cover of the case L was ground to fit against the lower part, and a thin coating of vacuum wax between the surfaces gave an excellent joint. A platinum wire for the electrical connection to the solenoid in the fall box was sealed through the glass tubing outside the case, the return circuit being through the metal parts of the apparatus, and its supports W,W.

Pressure readings were taken by means of a cathetometer on a closed-arm mercury manometer, and, when the pressure was low enough, on a McLeod gauge with a small factor.

Glass stopcocks served to isolate the apparatus from the pumps.

7. Method of Conducting an Experiment.

The following operations were necessary in preparing the apparatus for an experiment. The chronograph driving belt was put in place and the speed of rotation of the plate K, or wheel Q, was adjusted to a value depending on the pressure to be used. A complete rotation of Q occupied from 20 to 1200 seconds. The mercury cups T and the wheel Q were removed from N and replaced after a paper disc R had been fastened to Q. The index V and shield plate H were taken out and the glass plate K was wiped free from spheres. The armature O was brought into position over the magnet P by sending a current through the latter. The fall tube B, plate E, and fall box A, were put in place. The wheel Q was turned through a small angle and the plate K was watched through the window F to see that it moved freely. The window was then covered by a metal plate and the current through P was interrupted. The double connector for the solenoid circuit was put in place, and the fall box was arranged as in drawing VII or VIII for releasing spheres. The mechanism was tested by sending a current through the solenoid. This operation also supplied a check on the time record, for, since the plate K was at rest, the spheres which fell at this time should be at zero on the time record. The fall box was again arranged as before and the cover put in place. A dish containing P_2O_5 was placed inside the apparatus and the cover put in place. The drying agent was considered necessary, although experiments at atmospheric pressure showed the difference in viscosity due to a small amount of water vapor was not appreciable. At low pressures the same amount of vapor, forming a larger fraction of the whole gas, might change the viscosity considerably, since its viscosity is much lower than that of air. The pen was adjusted so that it would mark on the paper disc R.

The following operations were necessary in performing the experiment. They were never begun until the apparatus had been left to itself for at least three hours to allow it to come to a steady temperature. The pendulum was set swinging, and the pen was inspected to see that ink would flow freely. The chronograph motor was brought up to a constant speed. The circuit through the magnet P, and that including the pen and pendulum were closed, and by throwing a single lever the solenoid circuit was closed, releasing the spheres, and the wheel Q was put in gear with the chronograph, commencing to rotate at a uniform speed. When it had turned through nearly 360° the motor and time recording circuits were broken, and when the wheel Q had come to rest the end position of the pen was marked, and the pen lifted from the paper. The pressure was measured, and then air was slowly admitted until atmospheric pressure was reached.

The apparatus was prepared for finding and measuring the spheres by removing the cover, the drying material, the fall tube, fall box, and plate E, and putting in the shield plate H. The index V and mirror Y were placed as already explained, and the belt was taken off the chronograph to permit rotation of Q by hand. The measuring microscope was placed in guides which permitted it to move only along a radius of the plate K 180° from that passing under the fall tube. The distance from the center of the plate was controlled by a slow motion screw. With the microscope focussed on the surface of K the wheel Q and plate K were very slowly turned by hand through suitable gearing, and the spheres were measured as found, each being placed near the center of the field before measuring. The current for P was interrupted during measurement, to prevent creeping of the plate K. After surveying the whole of one strip or zone the microscope was moved radially the width of the field, and the process was repeated until enough spheres had been measured to show the variation of time of fall with radius. The radii for an experiment were not computed from the plate micrometer readings until all had been measured. The total number of spheres that fell to the plate was less than three hundred, so that no retarding effect on the velocity due to mutal action occurred.

IV. Results of Experiments.

In the tables and curves, I to XVI are given the results of the experiments, arranged in order of decreasing pressure. The quantities tabulated and plotted are the radii in 10^{-4} cm., and the reciprocals of the terminal velocities, that is, the times in seconds required to fall one cm. at this velocity. On the curves the radii are ordinates, and the times, abscissae. A small circle is drawn for each sphere measured.

The method of release shown in drawing VIII, figure 3, was found the most satisfactory of about seven methods, probably because the paper used closed the top of the fall tube throughout the experiment, and so no air currents set up by the movement of parts of the mechanism in the fall box could disturb the air in the tube. Large spheres did not, however, adhere to the paper, and so were not obtained by this method. The results obtained at atmospheric pressure using this method are given in table and curve I, except that the spheres for which a exceeds .0013 , were obtained by the method shown in drawing VII, figure 3. The results at atmospheric pressure using other methods than the one shown in drawing VIII are contained in table and curve II. In neither of these curves are any spheres shown which fell in the first twentieth of the period of a complete rotation of the wheel Q. The largest spheres shown are from experiments where the time of a rotation was 20 seconds, the smaller ones from experiments where the time was from 200 to 1200 seconds. The error in time measurements is thus kept small throughout the whole range.

The range of sizes used in all the experiments was from a = .0025 cm. to a = .00012 cm., and the range of pressures, from atmospheric pressure (740 mm. of mercury) down to .32 mm. of mercury. The total time of fall for the various sizes and pressures varied from .74 sec. to 1072 sec.

The value of the constant B in the formula was computed for each sphere in experiments VI, X, XIII, and XV, at pressures of 8.28, 2.11, .69, and .34 mm. of mercury, respectively. The average values of B for the four experiments were .00724, .00724, .00826, and .00729,* giving a mean value

$$B = .00753, \text{ corresponding to } A = 1.00 .$$

Values of the time per cm. were computed using this

* The close agreement of the first, second, and fourth values is probably accidental, since the limits of error in the time measurement and in the measurement of pressure are several per cent, that in the latter not being reduced in taking the average.

value of B, for each pressure used, and for the following values of the radius:-

.004	.0006
.0035	.0005
.003	.0004
.0025	.0003
.002	.00025
.0017	.0002
.0014	.00018
.0012	.00015
.001	.00014
.0008	.00013
	.00012

These values are plotted on curves I, and III to XVI as small crosses, and it will be seen that the circles representing observed spheres agree well with a line passing through these crosses. An idea of the great amount of variation in time of fall will be obtained by referring to curve I. The black dot near the left of the figure represents the time of falling one cm. for a sphere of radius .00014 cm., at a pressure of .32 mm. of mercury. At atmospheric pressure this sphere would fall in a time shown by the last cross at the right of the figure. The value of the term $\frac{B}{ap}$ in the formula varies from .003 for a = .004 cm., p = 740 mm., tp 196 for a = .00012 cm., p = .32 mm. The agreement of the experiments with the formula throughout the entire range, is excellent.

V. Spores.

Lycopodium spores, which fell at atmospheric pressure only about half as fast as Stokes's formula required, were found at lower pressures to maintain the same ratio of observed velocities to velocities computed by the formula which held at all pressures for wax spheres. This means that the density, which is the only measured quantity appearing in the formula in the first degree, was very incorrect. The density determined by refined volumenometer measurements, was 1.175 . This may have been the density of a solid shell surrounding an air space, or the density of a spongy mass containing many air spaces. The average density, meaning by this the quotient of the mass of a spore by the volume of a sphere having its mean radius, would have to be about .6 to make the spore behave as it does. In any event, turbulent motion of the air can no longer be held responsible for the variations observed at atmospheric pressure. It should be stated here that no evidence of a hollow or spongy structure of the spores has been observed, apart from the discrepancies in terminal velocity. Experiments with the other spores used before were not attempted, because the condition of the material appeared to have changed with time, and because the much smaller velocities to be measured would have made the constancy of temperature in the fall tube less certain.

VI. Summary and Conclusion.

The formula

$$V = \frac{2a^2 g(\sigma-\rho)}{9\mu}\left(1+A\frac{l}{a}\right) ,$$

in which A is a constant, expresses the terminal velocity of fall of small solid spheres in air throughout a wide range of radius and pressure. The constant A appearing in this formula is found to have a theoretical value depending on the assumption, consistent with the kinetic theory of gases, which is made concerning the mode of impact of the gas molecules on the surface of the sphere. Assuming elastic impacts the value of A is 1.5 ; assuming inelastic impacts and the same distribution of component velocities in the emerging as in the impinging molecules, the value of A is 1.2 ; assuming inelastic impacts and normal emergence of all impinging molecules, the value of A is 1.05 ; by experiment, the value of A is 1.0 .

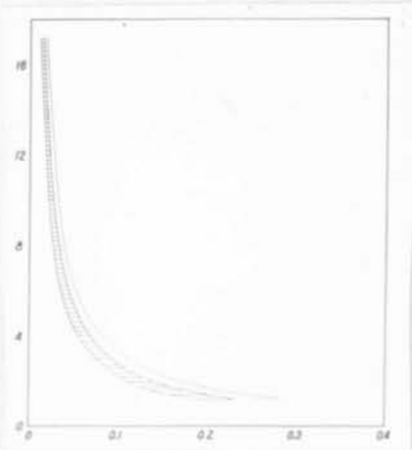


Figure 4.

The differences in computed velocities produced by using one or another value of A can be seen by referring to figure 4. Three curves are drawn, showing the variation of the time required to fall one cm. (abscissae) with the variation of the radius in $cm \times 10^{-4}$ (ordinates), at a pressure of .32 mm. of mercury, and at a temperature of 20° C. The curve at the left, and the one connected to it by horizontal lines are drawn for $A = 1.5$ and $A = 1.2$, respectively. The curve at the right is drawn for $A = 1.05$, the value corresponding to the assumption of normal emergence of the impinging gas molecules. Curve XVI, which shows the experimental results at this pressure agrees almost exactly with this curve. At atmospheric pressure the differences are not detectable by the method used, for the range of sizes obtained. This will be seen by comparing curves I and II. The crosses in II represent times computed by assuming $A = 0$ (Stokes's formula), those in I, times computed by assuming $A = 1.0$. The difference amounts to 7 per cent in time for the smallest sphere shown, being less for all larger spheres, and the additional difference in case $A = 1.5$ would be about 3 per cent, again for the smallest sphere shown. An error of 1.5 per cent in the measurement of the radius of such a sphere would entirely mask this difference in time of fall.

The only other experimental test of the formula derived by Cunningham is given by Professor R. A. Millikan[9], who found A = .815 . This value is determined from experiments on spheres of different sizes, but all at one pressure, namely, atmospheric. The method is somewhat indirect, since the radii of the spheres are computed from the time of fall, and were not actually measured. The material used was a liquid, and internal eddies in the drops may cause a consumption of energy, and consequently reduce the value of A.

The close agreement of the experiments with the formula derived for the assumption of normal emergence makes it probable that a gas molecule impinging on a solid is entangled in the surface layer of molecules, and emerges again after a number of collisions with these molecules, its direction of emergence being generally nearly normal to the surface. The experiments do not indicate whether this penetrable layer is composed of the same kind of mol-

---

[9] Science, Vol. XXXII., p. 446.

ecules as the rest of the solid, or is a condensed layer of the gas, but only show that the distances between the molecules in it are small compared to the distances in a gas under normal conditions of temperature and pressure.

I take great pleasure in thanking Professor John Zeleny for his continuous interest in the progress of the work, and for his many valuable suggestions on the solution of experimental and theoretical difficulties.

Explanation of tables.

p is the pressure in mm. of mercury.

t is the temperature in °C.

s is the length of fall.

T_m is the maximum time of fall measurable at the speed of rotation used.

a is the radius of the sphere.

$1/V$ is the time required to fall one cm. at the terminal velocity of fall, V.

Table I. Experiment 1.

p 736. t 20°.6 s 30.9 T_m 20

a	1/V
.002293	.1929
.002170	.1995
.001846	.2404
.001746	.297
.001461	.419
.001382	.437
.001369	.481
.001326	.474
.001308	.518

Experiment 2.

p 749. t 18°.3 s 30.96 T_m 226

a	1/V
.001369	.379
.001077	.600
.000906	.934
.000898	.934
.000757	1.365
.000719	1.524
.000696	1.566
.000679	1.718
.000674	1.653
.000653	1.748
.000637	1.916
.000608	2.169
.000604	2.088
.000585	2.270
.000580	2.189
.000560	2.296
.000541	2.45
.000541	2.70
.000522	2.56

Table I. Experiment 2. (con.)

.000505	2.86
.000492	3.36
.000466	3.47
.000461	3.75
.000453	3.65
.000449	3.59
.000433	4.10
.000432	3.86
.000427	4.18
.000426	3.96
.000417	4.18
.000416	4.56
.000407	4.51
.000393	4.86
.000385	4.97
.000385	5.40
.000382	4.52
.000373	5.45
.000368	5.49
.000355	5.52
.000350	5.88
.000346	6.10
.000346	6.67
.000339	6.63
.000335	6.72
.000334	6.79

Experiment 3.

p 740. t 19.7 s 30.96 T 1080

a	1/V
.000704	1.492
.000571	2.335
.000562	2.205
.000518	2.854
.000502	2.984
.000490	3.081

Table I. Experiment 3. (con.)

,000473	3.31
.000413	4.44
.000378	5.52
.000370	6.03
.000361	6.26
.000351	6.52
.000327	8.17
.000317	8.69
.000303	9.63
.000287	11.03
.000274	10.86
.000273	13.26
.000238	15.18
.000238	15.79
.000218	18.49
.000213	17.96
.000210	20.24
.000192	23.22
.000191	27.64
.000162	34.76

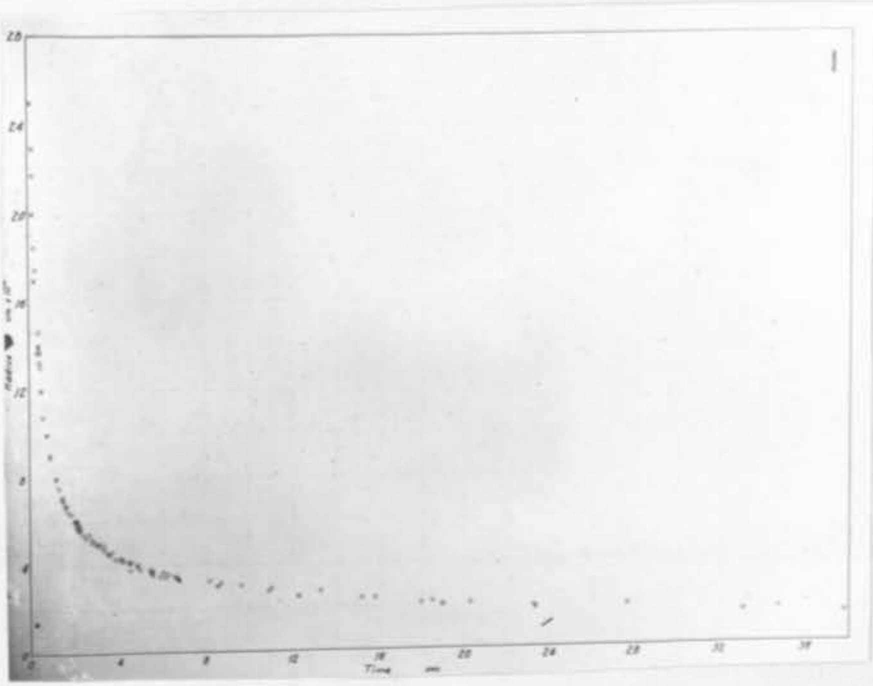


Table II. Experiment 1.

p 753. t 19°.7 s 30.9 T_m 217

a	1/V
.001536	.325
.001482	.351
.001441	.325
.001440	.358
.001410	.358
.001397	.387
.001365	.410
.001268	.462
.001113	.455
.000825	1.433
.000707	1.378
.000676	1.722
.000649	1.979
.000644	1.992
.000632	1.800
.000552	2.405
.000465	3.670
.000462	3.718
.000441	3.847
.000430	3.784
.000417	4.65
.000323	7.04

Experiment 2.

p 735. t 20°.1 s 30.9 T_m 230

a	1/V
.001423	.390
.001277	.507
.001276	.471
.001166	.494
.001029	.686
.000840	1.011

Table II. Experiment 2. (con.)

.000723	1.485
.000617	1.914
.000563	2.135
.000522	2.619
.000471	3.096
.000393	4.10
.000386	4.40
.000344	5.11
.000342	5.70
.000340	5.92
.000324	5.14
.000323	7.19

Experiment 3.

p 740. t 20.°6 C. s 30.9 T_ 54

a	1/V
.001191	.520
.001149	.397
.000664	1.732

Experiment 4.

Same as Table I. Experiment 1.

Experiment 5.

p 734. t 21.°0 C. s 30.9 T_ 894

a	1/V
.000575	2.046
.000563	2.404
.000556	2.631
.000547	2.566
.000486	3.23
.000423	4.32
.000374	5.39

Table II. Experiment 5. (con.)

.000327	6.66
.000318	9.23
.000307	8.64
.000306	7.76
.000279	10.07
.000254	14.13
.000250	14.39
.000247	13.02
.000180	20.99
.000179	22.74
.000164	24.56

Experiment 6.

p 740. t 20°.7 C. s 30.9 T_m 191

a	1/V
.001657	.344
.001623	.370
.001510	.370
.001144	.659
.001127	.637
.000853	1.017
.000535	2.83
.000396	5.13
.000352	6.06

Experiment 7.

p 740. t 20°.7 C. s 31.1 T_m 197

a	1/V
.001473	.342
.001117	.571
.001035	.726
.000789	1.221

Table II. Experiment 8.

p 742. t 19.°4 s 31.4 T_m 237

a	1/V
.001243	.441
.001185	.460
.001003	.758
.000903	.988
.000874	1.097
.000873	.972
.000863	.988
.000807	1.198
.000733	1.500
.000726	1.429
.000710	1.506
.000709	1.547
.000625	1.739

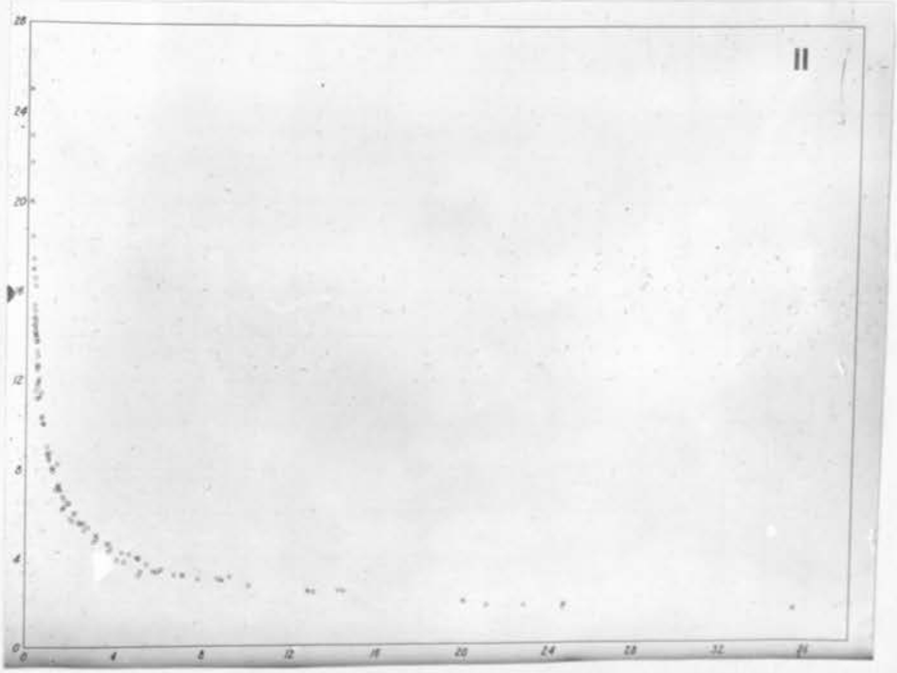


Table III. Experiment 1.

p 100.4 t 21°.0 s 30.9 T_212

a	1/v
.002660	.1462
.002177	.2046
.002069	.2795
.001918	.296
.001854	.221
.001840	.260
.001531	.318
.001293	.482
.001207	.497
.001176	.520
.001068	.695
.000992	.754
.000833	1.170
.000793	1.011
.000712	1.436
.000641	1.781
.000543	2.64
.000537	2.38
.000529	2.52
.000497	2.95
.000497	3.25
.000440	3.49
.000438	3.87
.000320	6.76
.000313	6.60
.000302	6.88

Experiment 2.

p 100.6 t 20°.6 s 30.9 T_ 17

a	1/v
.002981	.0842
.002063	.1929

Table III. Experiment 2. (con.)

.001732	.276
.001648	.300
.001623	.307
.001289	.409
.001145	.556

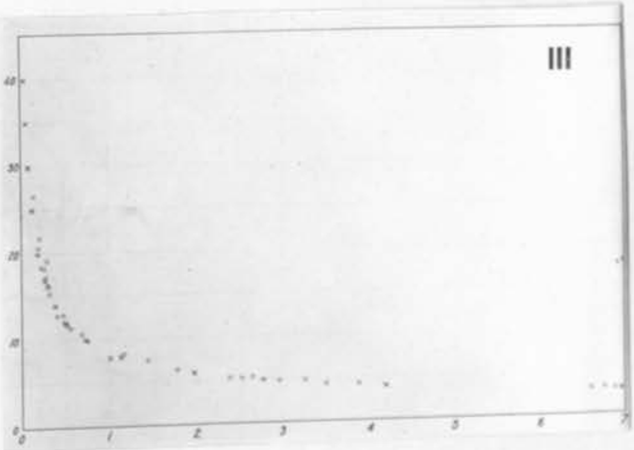


Table IV. Experiment 1.

p 33.64 t 21°.3 s 30.9 T_{m} 80

a	1/V
.002479	.1137
.002380	.1137
.001933	.1884
.001792	.2113
.001545	.3217
.001458	.354
.001240	.539
.000910	.844
.000898	.900
.000850	1.075
.000763	1.112
.000756	1.254
.000716	1.364
.000639	1.803
.000488	2.294

Experiment 2.

p 34.85 t 21°.2 s 30.9 T_{m} 186

a	1/V
.003715	.0715
.002989	.1040
.002642	.1040
.002591	.1234
.001506	.351
.001347	.429
.001199	.487
.001109	.604
.001095	.617
.001013	.676
.000975	.669
.000926	.767
.000909	.786

Table IV. Experiment 2. (con.)

.000908	.812
.000808	1.027
.000807	1.053
.000737	1.170
.000635	1.527
.000634	1.501
.000623	1.689
.000532	2.053
.000516	2.339
.000487	2.385
.000460	2.781
.000380	4.14
.000331	5.10
.000330	4.71

Experiment 3.

p 33.30 t 20°.5 s 30.9 T 18

a	1/V
.003544	.0874
.003291	.0971
.002442	.1312
.001603	.291
.001216	.539
.001175	.565

Experiment 4.

p 34.77 t 22°.1 s 30.9 T 886

a	1/V
.000666	1.299
.000597	1.462
.000588	1.364
.000545	1.721
.000521	1.656

Table IV. Experiment 4. (con.)

.000515	1.754
.000460	2.83
.000458	2.21
.000439	2.44
.000428	2.79
.000419	2.83
.000408	2.60
.000395	2.99
.000344	4.16
.000299	5.17
.000296	5.72

Experiment 5.

p 33.58 t 20.9 s 30.9 T_m 20

a	1/V
.001749	.268
.001531	.314
.001413	.350
.001154	.538
.000989	.648

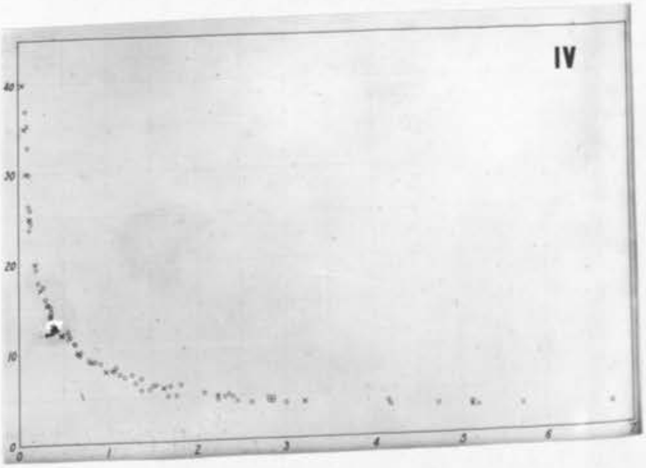


Table V.

p 8.55 t 20°.5 s 30.96 T 177

a	1/V
.000865	.574
.000444	1.488
.000424	1.527
.000307	2.45
.000234	3.32
.000152	4.80
.000121	5.33

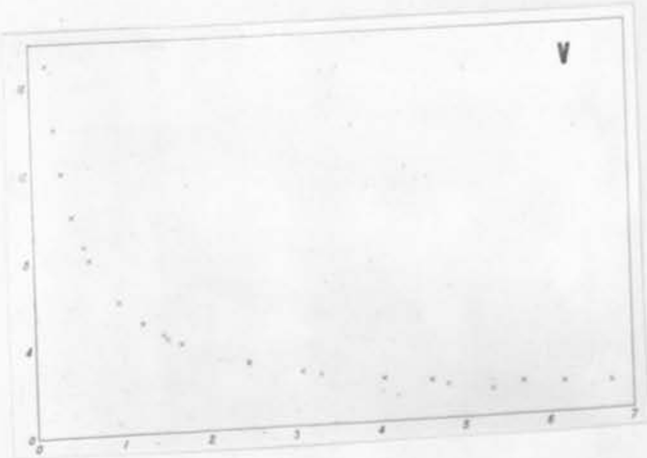


Table VI.

p 8.28 t 21°.3 s 30.96 T_ 172

a	1/V
.001769	.1427
.001411	.1978
.001216	.292
.001141	.353
.000819	.584
.000766	.652
.000752	.712
.000599	.960
.000569	1.067
.000518	1.183
.000517	1.118
.000477	1.287
.000430	1.472
.000429	1.517
.000399	1.686
.000343	1.861
.000316	2.250
.000303	2.445
.000287	2.57
.000263	2.86
.000257	2.94
.000220	3.67
.000219	3.71
.000204	4.10
.000175	4.64
.000175	5.18
.000169	4.79
.000119	5.54

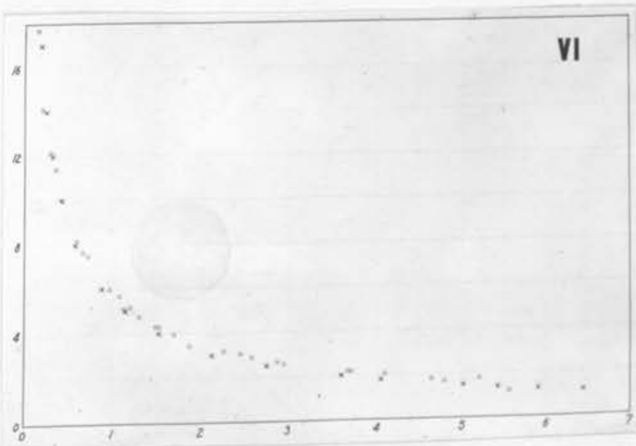

VI

Table VII.

p 8.14 t 21°.5 s 30.96 T. 176

a	1/V
.000961	.506
.000934	.499
.000812	.606
.000733	.710
.000602	.886
.000571	.953
.000474	1.271
.000438	1.427
.000428	1.404
.000393	1.686
.000344	1.916
.000328	2.098
.000323	2.146
.000318	2.053
.000318	2.098
.000315	2.186
.000294	2.455
.000287	2.494
.000271	2.624
.000264	2.77
.000260	2.85
.000256	2.87
.000255	3.02
.000253	2.96
.000232	3.16
.000223	3.35
.000202	3.84
.000197	4.05
.000182	4.26
.000170	4.68
.000167	4.84
.000162	5.03
.000159	5.15
.000157	5.18
.000151	5.14
.000146	5.74

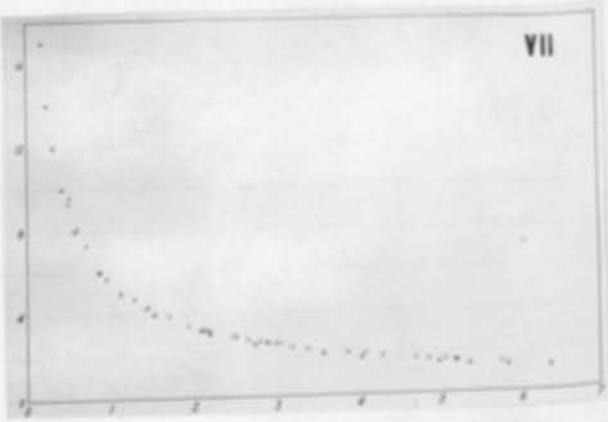

VII

Table VIII. Experiment 1.

p 7.70 t 20°.9 s 30.9 T_m 184

a	1/V
.002090	.1787
.002038	.1787
.001375	.276
.001126	.370
.001065	.491
.001059	.445
.001044	.406
.000922	.478
.000916	.500
.000873	.533
.000784	.630
.000764	.650
.000704	.689
.000701	.712
.000688	.731
.000621	.896
.000555	.965
.000525	1.104
.000514	1.072
.000510	1.153
.000507	1.004
.000494	1.134
.000466	1.495
.000382	1.589
.000327	1.949
.000316	1.888
.000307	2.066
.000306	2.129
.000274	2.556
.000257	2.550
.000210	3.17
.000179	3.12

Table VIII. Experiment 2.

p 7.72 t 20°.8 s 30.9 T_w 20

a	1/V
.001504	.240
.001195	.338
.000934	.474
.000857	.556

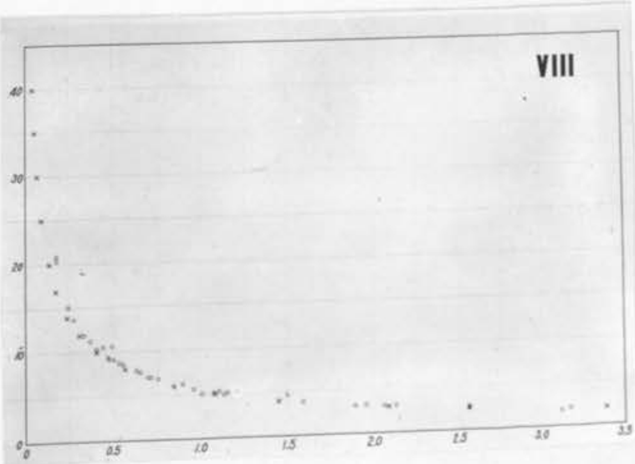


Table IX. Experiment 1.

p 3.10 t 20°.0 s 30.9 T_m 20

a	1/V
.001534	.1588
.001303	.1946
.001301	.1751
.000981	.237
.000917	.270
.000883	.270
.000797	.305
.000753	.304
.000734	.383
.000707	.351
.000671	.390

Experiment 2.

p 3.14 t 20°.2 s 30.9 T_m 20

a	1/V
.002042	.0809
.002027	.0923
.002007	.0939
.001966	.1134
.001790	.1150
.001775	.1118
.001372	.1556
.001365	.1540
.001327	.1703
.001268	.1832
.001240	.1914
.001145	.1654
.001106	.2012
.001038	.2109
.000930	.2275
.000892	.2616
.000890	.289

Table IX. Experiment 2. (con.)

.000709	.341
.000699	.390
.000687	.419
.000658	.401
.000609	.418
.000565	.460
.000538	.487
.000502	.496
.000456	.565
.000321	.625

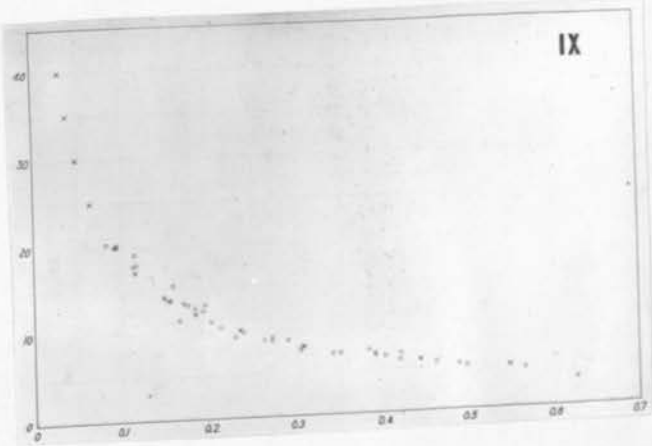


Table X.

p 2.11 t 21.3 s 30.96 T_m 149

a	1/V
.001168	.1167
.001145	.1264
.001095	.1783
.000831	.2076
.000805	.2205
.000777	.2594
.000711	.2756
.000590	.341
.000544	.360
.000516	.376
.000499	.409
.000425	.435
.000423	.470
.000400	.525
.000369	.587
.000321	.684
.000308	.720
.000269	.817
.000261	.904
.000240	.992
.000226	1.099
.000177	1.439
.000176	1.446
.000135	1.702
.000132	2.082

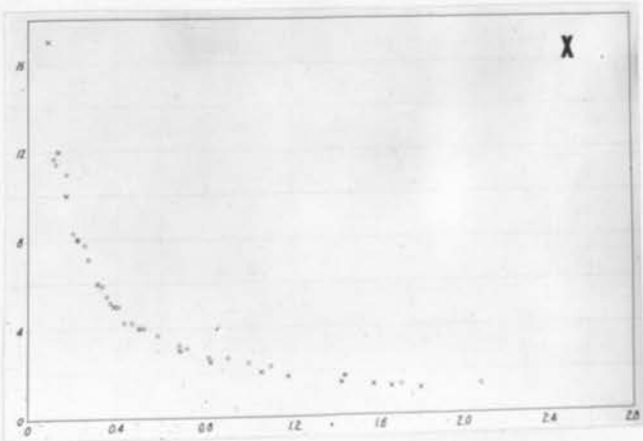

X
16
12
8
4
0
0
0.4
0.8
1.2
1.6
2.0
2.4
2.8

Table XI. Experiment 1.

p 2.01 t 20°.6 s 30.9 T_m 20

a	1/V
.002396	.0341
.002358	.0387

Experiment 2.

p 2.01 t 20°.5 s 30.9 T_m 19

a	1/V
.001357	.1208
.001213	.1622
.000940	.1888
.000931	.1969
.000768	.2278
.000764	.2274
.000670	.2596
.000626	.275
.000624	.304
.000569	.317
.000515	.322
.000510	.345

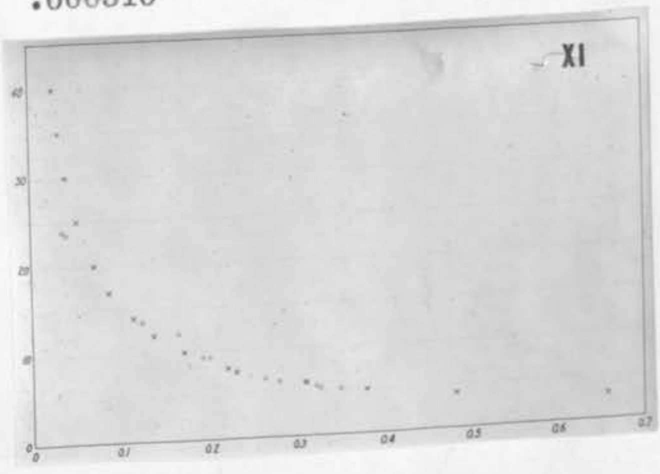


Table XII.

p 1.28 t 20°.7 s 30.9 T_m 18

a	1/V
.002222	.0426
.001616	.0647
.001533	.0672
.001422	.0738
.001323	.0838
.001320	.0844
.001164	.0893
.001049	.1082
.001033	.0910
.000883	.1248
.000808	.1270
.000752	.1494
.000623	.1595
.000588	.1914
.000554	.2034
.000512	.259
.000408	.280
.000401	.324
.000394	.321
.000391	.287
.000364	.311
.000340	.348
.000304	.340
.000294	.399
.000293	.420
.000281	.392
.000245	.457
.000233	.548

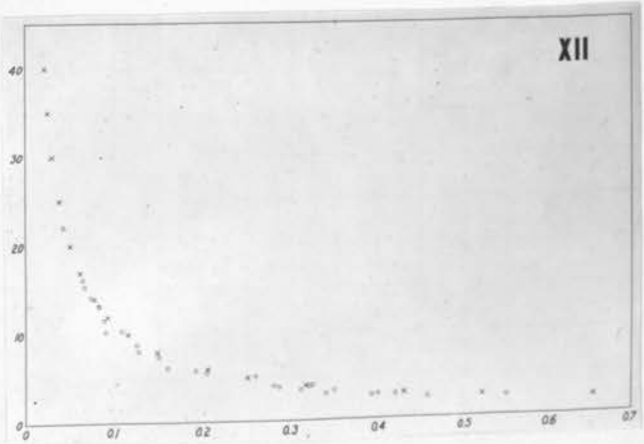

XII
40
30
20
10
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7

Table XIII.

p .69 t 21°.8 s 30.96 T_{m} 22

a	1/V
.001380	.0405
.001361	.0435
.000964	.0674
.000793	.0830
.000678	.1141
.000632	.0970
.000589	.1083
.000507	.1196
.000463	.1472
.000414	.1537
.000401	.1595
.000396	.1579
.000391	.1628
.000385	.1754
.000349	.1942
.000319	.2062
.000309	.2254
.000284	.237
.000257	.263
.000239	.282
.000222	.313
.000207	.347
.000204	.343
.000158	.448
.000133	.430
.000125	.477
.000115	.482

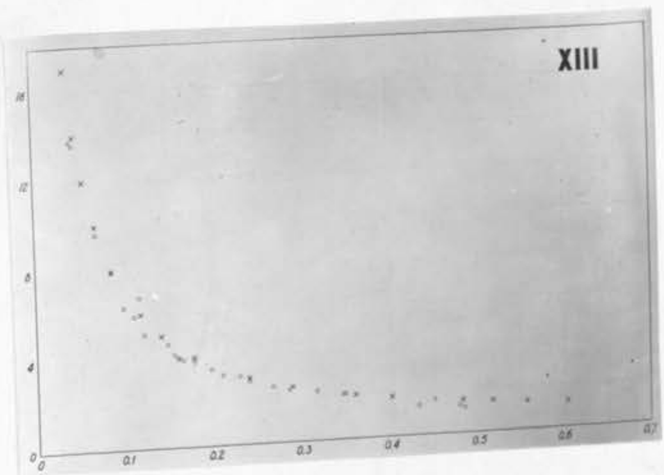

XIII
16
12
8
4
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7

Table XIV.

p .40 t 21°.1 s 30.96 T_m 16

a	1/V
.000975	.0493
.000791	.0551
.000654	.0662
.000639	.0688
.000595	.0759
.000556	.0850
.000474	.0856
.000429	.0921
.000384	.1157
.000367	.1180
.000351	.1232
.000351	.1252
.000344	.1274
.000323	.1268
.000320	.1271
.000305	.1378
.000261	.1579
.000259	.1670
.000256	.1680
.000256	.1718
.000243	.1721
.000241	.1932
.000228	.1848
.000221	.2056
.000219	.1987
.000219	.2033
.000211	.227
.000200	.242
.000188	.265
.000186	.258
.000162	.280
.000156	.316
.000139	.363
.000133	.355

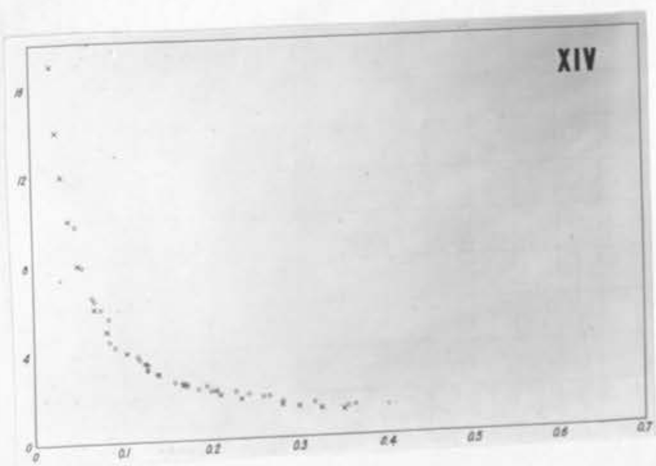

XIV
16
12
8
4
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7

Table XV.

p .34 t 21°.3 s 30.96 T_{∞} 16

a	1/V
,001501	.0240
.000898	.0379
.000736	.0412
.000684	.0454
.000624	.0538
.000556	.0700
.000425	.0788
.000401	.0885
.000386	.0862
.000383	.0898
.000340	.1161
.000332	.1154
.000319	.1148
.000271	.1540
.000226	.1987
.000213	.1922
.000200	.1974
.000173	.2376
.000119	.306

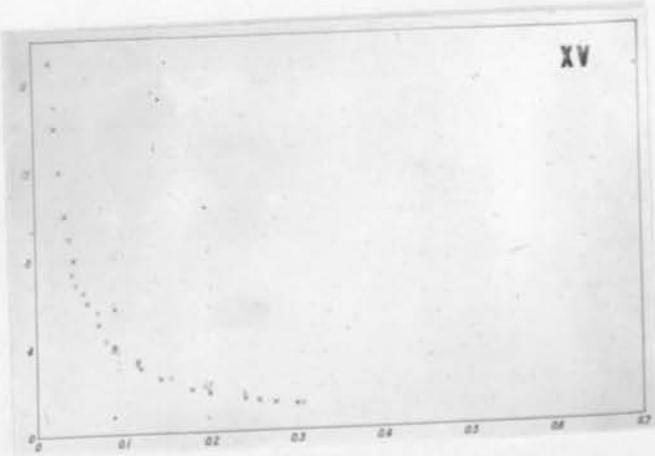
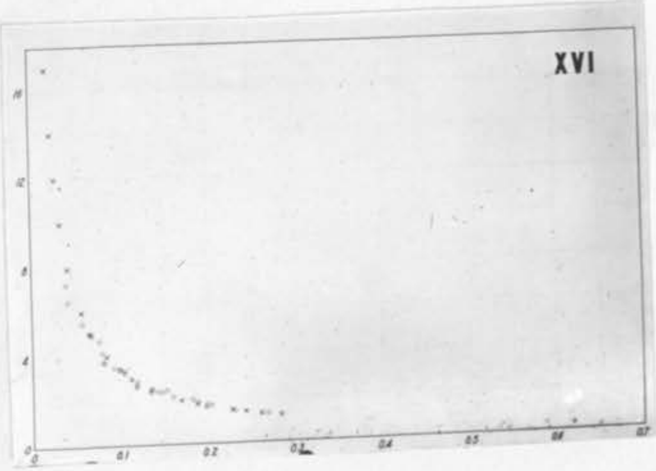


Table XVI.

T. 17

p .32 t 21°.3 s 30.96

a	1/V
.001163	.0341
.000730	.0379
.000647	.0405
.000548	.0564
.000501	.0638
.000494	.0668
.000468	.0752
.000415	.0788
.000381	.0798
.000379	.0814
.000357	.0914
.000349	.0992
.000347	.0973
.000340	.1028
.000327	.1031
.000289	.1177
.000273	.1167
.000263	.1161
.000253	.1346
.000244	.1524
.000242	.1417
.000239	.1333
.000239	.1472
.000216	.1598
.000202	.1816
.000190	.1871
.000178	.1968
.000172	.2024
.000163	.1961
.000142	.228
.000127	.266


XVI
0
4
8
12
0.1
0.2
0.3
0.4
0.5
0.6
0.7